# Lecture 3:

Note Title 26/03/2021

**Yesterday:** $|\psi\rangle = a|0\rangle + b|1\rangle$

$= c|+\rangle + d|-\rangle$

**inner products**

$\langle\psi| = (|\psi\rangle)^\dagger$

$= a^*\langle 0| + b^*\langle 1|$

$\langle 0| = (|0\rangle)^\dagger$

$= \begin{pmatrix}1\\0\end{pmatrix}^\dagger = \begin{pmatrix}1^*\\0^*\end{pmatrix}^\dagger = \begin{pmatrix}1\\0\end{pmatrix}^\dagger = (1\ \ 0)$

$\langle\psi|\psi\rangle = (a^*\langle 0| + b^*\langle 1|)(a|0\rangle + b|1\rangle)$

$= \underbrace{|a|^2}_{\to p(0)} + \underbrace{|b|^2}_{\to p(1)} = 1$

**Outer product**

$|1\rangle\langle 0|, \ \dots$

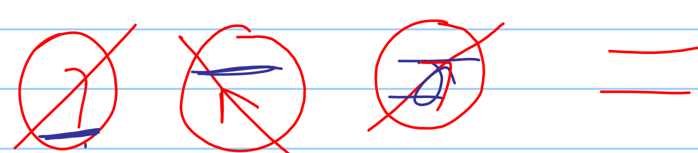
$= \begin{pmatrix}0\\1\end{pmatrix}(1\ \ 0)$

$= \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

$G = \sum a_{ij}|i\rangle\langle j| = \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}$

**Multipartite**

$|\psi_1\rangle_A \otimes |\psi_2\rangle_B \otimes |\psi_3\rangle_C$



A B C

$\boxed{|0\rangle_A \otimes |1\rangle_B \otimes |+\rangle_C}$

$|1\rangle_B \otimes |0\rangle_A \otimes |+\rangle_C$

$|0\rangle_2 \otimes |1\rangle_1 \otimes |+\rangle_0$

$|1\rangle \otimes |0\rangle \otimes |+\rangle$

$|0\rangle|1\rangle|+\rangle$

$= |0 1 +\rangle = |0\rangle_{(0)} \otimes |1\rangle_{(1)} \otimes |+\rangle_{(2)}$

**Bipartite**

$|a\rangle = a_1|0\rangle + a_2|1\rangle \; ; \; |b\rangle = b_1|0\rangle + b_2|1\rangle$

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

$(4 \times 1)$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \quad |0\rangle|0\rangle = |00\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|11\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad |00\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

**Basis**

- $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ are $\perp$

$$\langle 10 | 00 \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (0 \; 0 \; 1 \; 0) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 0 + 0 + 0 + 0 = 0$$

**Example!** $|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}}\left(|11\rangle + |01\rangle + |10\rangle\right)$

$$\{|++\rangle, |+-\rangle, |--\rangle, |-+\rangle\}$$

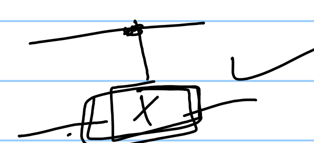
$$|+\rangle \otimes |-\rangle = \frac{1}{(\sqrt{2})^2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

- $$|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle = \begin{pmatrix}a_1\\a_2\end{pmatrix} \otimes \underline{\begin{pmatrix}b_1\\b_2\end{pmatrix} \otimes \begin{pmatrix}c_1\\c_2\end{pmatrix}}$$

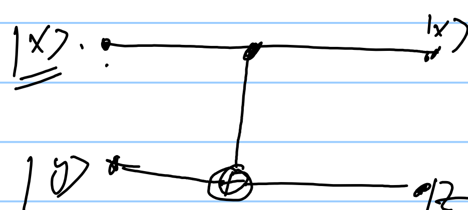
$$= \begin{pmatrix}a_1\\a_2\end{pmatrix} \otimes \begin{pmatrix}b_1c_1\\b_1c_2\\b_2c_1\\b_2c_2\end{pmatrix}$$

$$= \begin{pmatrix}\vdots\\\vdots\\\vdots\end{pmatrix} \quad 8\times 1$$

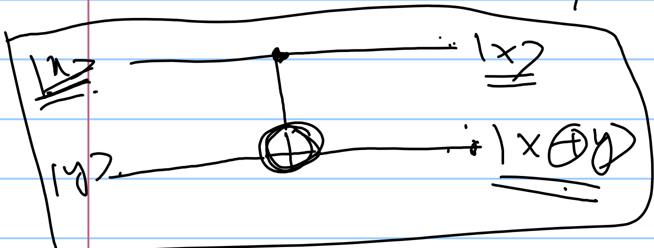
## Two qubit Gates




- CNOT




| $|x\rangle|y\rangle$ | $|x\rangle|z\rangle$ |
| --- | --- |
| $|0\rangle|0\rangle$ | $|0\rangle|0\rangle$ |
| $|0\rangle|1\rangle$ | $|0\rangle|1\rangle$ |
| $|1\rangle|0\rangle$ | $|1\rangle|1\rangle$ |
| $|1\rangle|1\rangle$ | $|1\rangle|0\rangle$ |

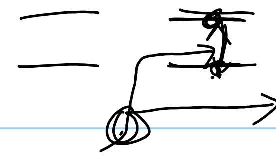



$$|z\rangle = |x \oplus y\rangle$$

$|x\rangle = |1\rangle$

$|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$C_x|y\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$



$C_x = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} |x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |y\rangle = |0\rangle \end{cases}$$

$C_x|x\rangle|y\rangle = C_x \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$

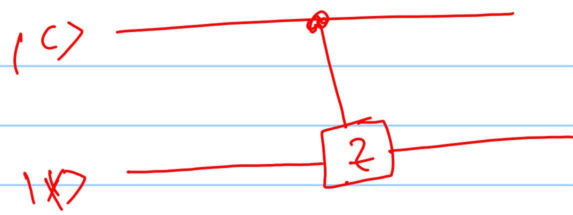
$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$C_x|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$

$= |+\rangle \otimes |1\rangle$

CZ



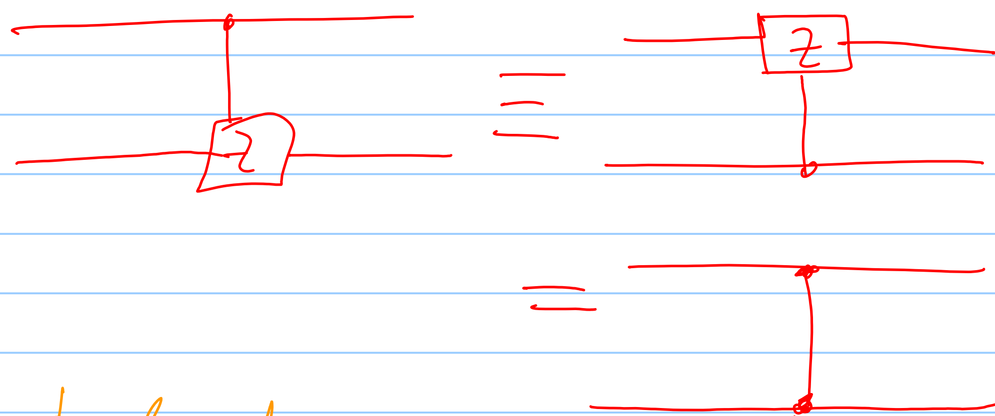

ZX if $|c\rangle = 1$

X if $|c\rangle = 0$

$|0\rangle \xrightarrow{Z} |0\rangle$

$|1\rangle \rightarrow -|1\rangle$

| $\lvert c\rangle$ | $\lvert x\rangle$ | | |
|---|---|---|---|
| $\lvert 0\rangle$ | $\lvert 0\rangle$ | $\longrightarrow$ | $\lvert 0\rangle \lvert 0\rangle$ |
| $\lvert 0\rangle$ | $\lvert 1\rangle$ | $\longrightarrow$ | $\lvert 0\rangle \lvert 1\rangle$ |
| $\lvert 1\rangle$ | $\lvert 0\rangle$ | $\longrightarrow$ | $\lvert 1\rangle \lvert 0\rangle$ |
| $\lvert 1\rangle$ | $\lvert 1\rangle$ | $\longrightarrow$ | $-\lvert 1\rangle \lvert 1\rangle$ |




## Entanglement:

$$|\psi\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B \quad \cancel{\otimes} \quad \cancel{\otimes}$$

$|0\rangle$

$|1\rangle$

$$\neq |\psi\rangle \otimes |\varphi\rangle \longrightarrow \text{entangled}$$

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |11\rangle\right)} \checkmark$$

$$\neq |\rangle \otimes |\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right) \quad \times$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)$$

$|\psi\rangle = |0\rangle \otimes |+\rangle$

# Maximally entangled states

## Bell state

$\langle \psi^{00} | \psi^{10} \rangle = 0$

$\langle 00|01\rangle = 0$

$\langle 00|10\rangle = 0$

$\vdots$

$$|\psi\rangle^{00} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$|\psi\rangle^{10} = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

$$|\psi\rangle^{01} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$

$$|\psi\rangle^{11} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$\Rightarrow$ pos

=

Basis $\underbrace{|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle}$

$\rightarrow |{+}{+}\rangle = |+\rangle \otimes |+\rangle$

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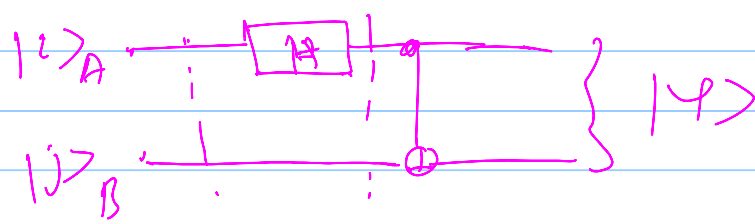



- $|i\rangle = |0\rangle, \quad |j\rangle = |0\rangle$

$\downarrow$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle\right)$$

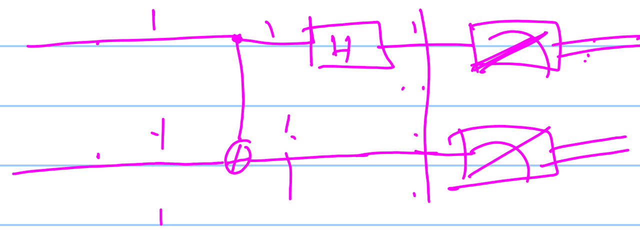
$$CX \Longrightarrow = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) = |\psi^{00}\rangle$$

- $|i\rangle = 0, \quad |j\rangle = |1\rangle$

$\Longrightarrow |\psi^{01}\rangle$

# Measurement in Bell basis

$|\psi\rangle$

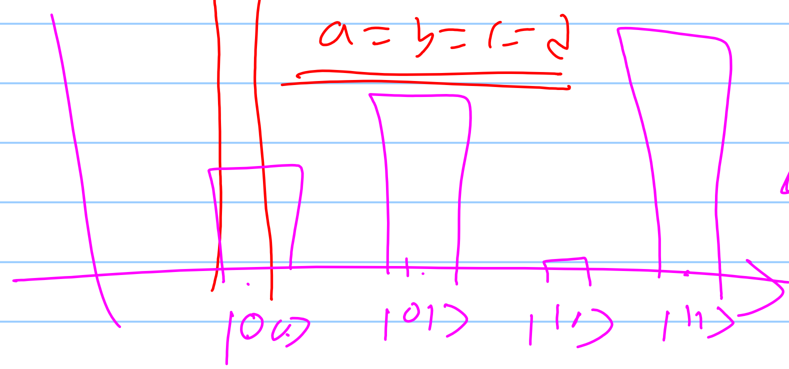


$$|\psi\rangle = a|\phi\rangle^{00} + b|\phi\rangle^{01} + c|\phi\rangle^{10} + d|\phi\rangle^{11}$$

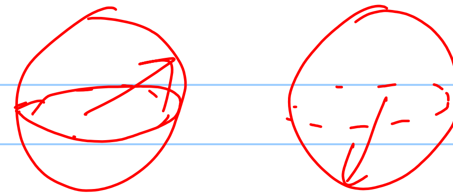
$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$|00\rangle$

$a = b = c = d$



$$|\psi_1\rangle \otimes |\psi_2\rangle$$

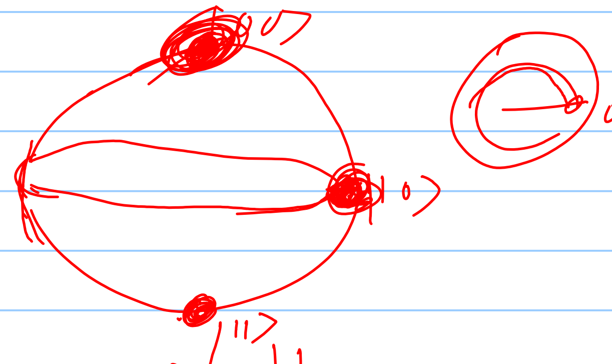


A B

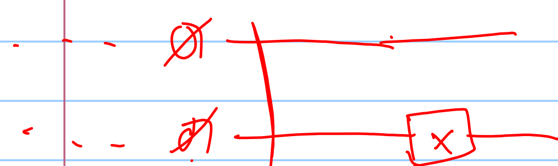
$$\underline{\underline{|\psi\rangle}}$$

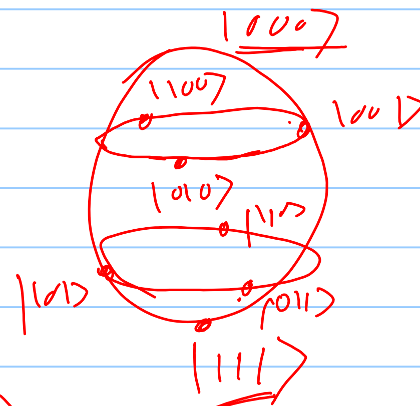
$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(3\,\underline{|00\rangle} + 2|10\rangle + \underline{|11\rangle}\right)$$

$$= \left(3e^{i60^\circ}\,\underline{|00\rangle} + 2i|10\rangle - |11\rangle\right)$$




Q-Sphere




$$O = \underline{1 \otimes X}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}}$$