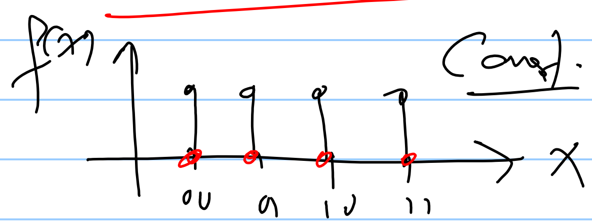
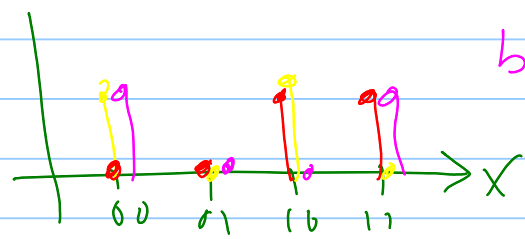
Lecture 4 DJ, BV Algorithms

Deutsch Jozsa Algorithm

Deutsch Problem:



$$f(x) = \begin{cases} \text{Const. } 0 \text{ or } 1 \\ \text{balanced} \end{cases}$$

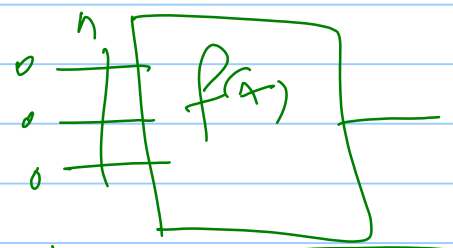


balanced function.

How many queries to find $f(x)$?

Classical:

000 → 0
001 → 0
010 → 0
⋮

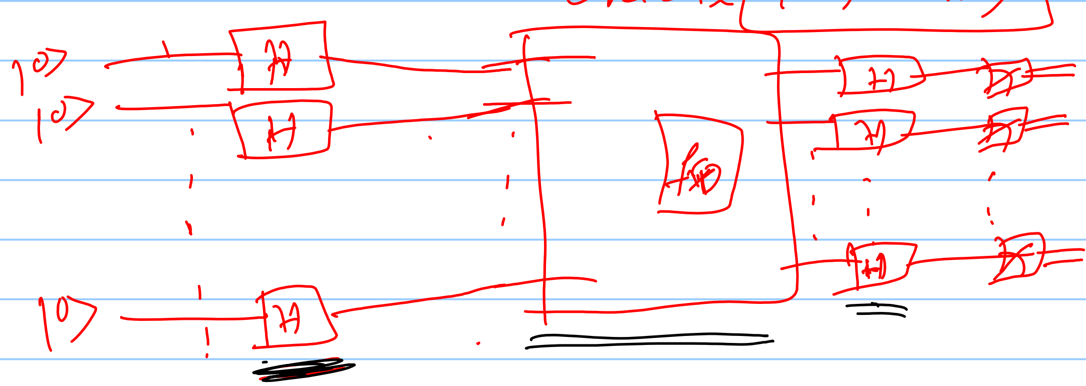


$N = 2^n$

$\boxed{\frac{N}{2} + 1 \text{ queries}}$

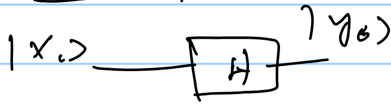
Q.C.

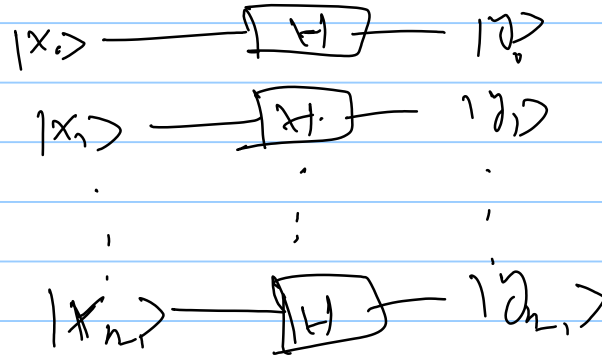
oracle $\boxed{(-1)^{f(x)} |x\rangle}$



Hadamard Transform:

Examples





$x_0 = 0$

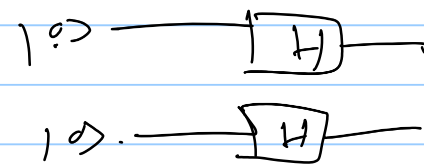
$$y_0 = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$|x_1\rangle = |1\rangle$

$$|y_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

---

input $|00\rangle$



$$|y_0 y_1\rangle = |y_0\rangle \otimes |y_1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|y_0 y_1\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

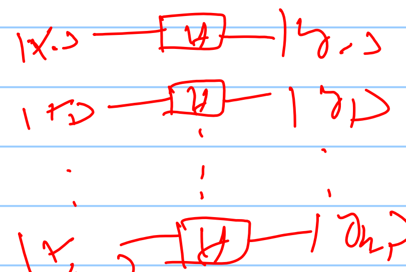
---

input $|01\rangle$

$$|y_0 y_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

General



$$|y\rangle = H^{\otimes n} |x\rangle$$

$$|y_0 y_1 \cdots y_{n-1}\rangle = H|x_0\rangle \otimes H|x_1\rangle \cdots \otimes H|x_{n-1}\rangle$$

$$\boxed{|y\rangle = H|x\rangle}$$

$$|y\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_1}|1\rangle\right)$$

$$\otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_{n-1}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_0=0}^{1} (-1)^{k_0 x_0} |k_0\rangle \otimes \sum_{k_1=0} (-1)^{k_1 x_1} |x_1\rangle$$

$$\otimes \cdots \otimes \sum_{k_{n-1}=0}^{1} (-1)^{k_{n-1} x_{n-1}} |k_{n-1}\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_0} \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{n-1}} (-1)^{k_0 x_0 + k_1 x_1 \cdots k_{n-1} x_{n-1}} |k_0 k_1 \cdots k_{n-1}\rangle$$

$$|y\rangle = H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_k (-1)^{k \cdot x} |k\rangle$$

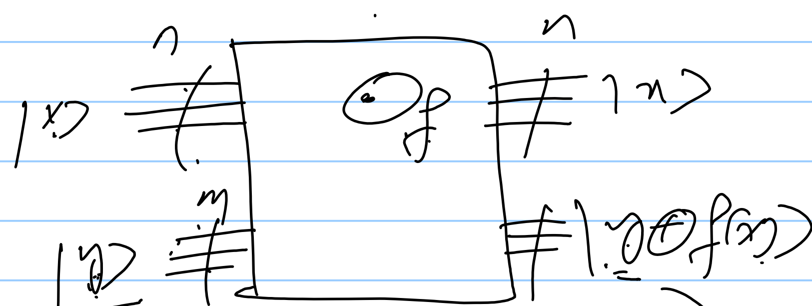
$$|x\rangle = |00\cdots0\rangle = |0\rangle$$

$$\boxed{H^{\otimes n}|0\rangle = \frac{1}{\sqrt{2^n}} \sum_k |k\rangle}$$

Oracle:

$0 \oplus 0 = 0$

$0 \oplus 1 = 1$

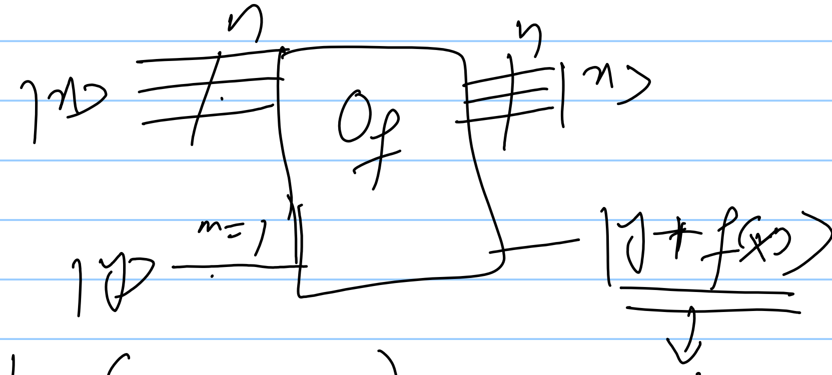


if $|y\rangle = 0$

$|f(x)\rangle$

Phase oracle:

$f(x): \{0,1\}^n \rightarrow \{0,1\}$



$$|y\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$|x\rangle|y\rangle$$

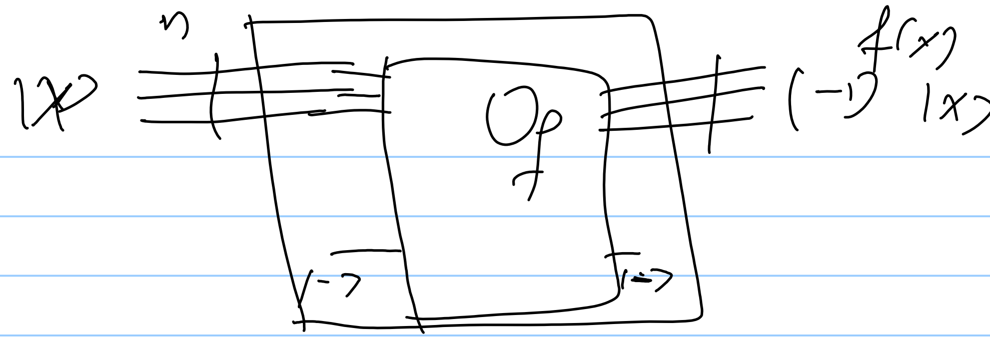
$$= |x\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$\Rightarrow \quad = |x\rangle \frac{1}{\sqrt{2}}\left(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

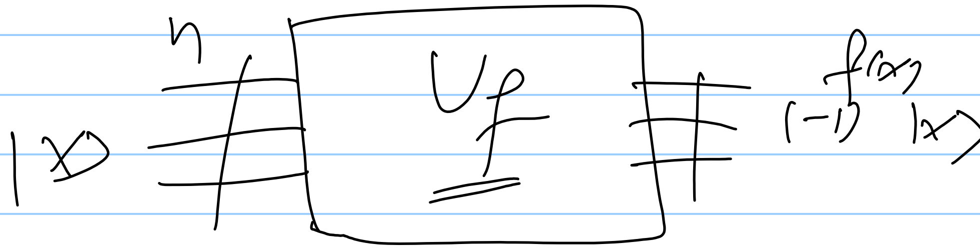
$$= |x\rangle \frac{1}{\sqrt{2}}\left(|f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

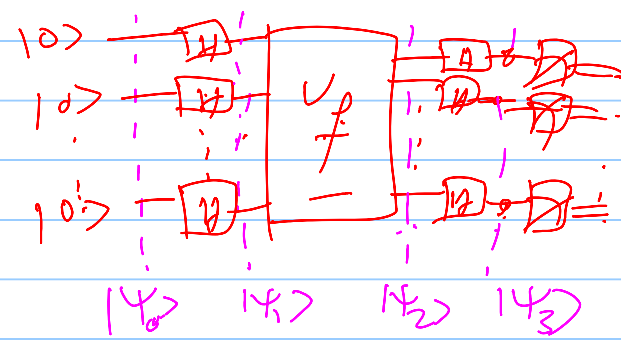
$$= \begin{cases} |x\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) & f(x) = 0 \\ = +|x\rangle|y\rangle & \\ |x\rangle \frac{1}{\sqrt{2}}\left(|1\rangle - |0\rangle\right) & f(x) = 1 \\ = -|x\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) & \\ = -|x\rangle|y\rangle & \end{cases}$$

$$|x\rangle|y\rangle \longrightarrow \underline{(-1)^{f(x)}}\,|x\rangle|y\rangle$$



Phase Oracle





(0) $|\psi_0\rangle = |00\cdots0\rangle$

$= |0\rangle^{\otimes n}$

(1) $|\psi_1\rangle = H^{\otimes n}|\psi_0\rangle$

$= \frac{1}{\sqrt{2^n}} \sum_k (-1)^{k\cdot x} |k\rangle$

$= \frac{1}{\sqrt{2^n}} \sum_k |k\rangle$

$N \propto 2^n$

(1)

(2) $|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_n U_f |k\rangle$

$= \frac{1}{\sqrt{2^n}} \sum_n (-1)^{f(k)} |k\rangle$

(3) $|\psi_3\rangle = H^{\otimes n}|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_k (-1)^{f(k)} H^{\otimes n} |k\rangle$

$= \frac{1}{\sqrt{2^n}} \sum_k (-1)^{f(k)} \underbrace{\frac{1}{\sqrt{2^n}} \sum_l (-1)^{l\cdot k} |l\rangle}$

$= \frac{1}{2^n} \sum_l \left(\sum_k (-1)^{f(k)+l\cdot k} \right) |l\rangle$

$$\boxed{|\psi_3\rangle = \frac{1}{2^n}\sum_{\ell} c_\ell |\ell\rangle}$$

$$c_\ell = \sum_k (-1)^{f(k)+\ell\cdot k}$$

Proof.

$$\psi_3(0) = |\langle 00\cdots 0|\psi_3\rangle|^2$$

$$= \left|\frac{1}{2^n}\sum_\ell c_\ell \langle 00\cdots 0|\ell\rangle\right|^2$$

$$= \left|\frac{1}{2^n} c_0\right|^2$$

$$= \left|\frac{1}{2^n}\sum_k (-1)^{f(k)+\cancel{k\cdot 0}^{\,0}}\right|^2$$

$$= \left|\frac{1}{2^n}\sum_{k=0}^{2^n-1} (-1)^{f(k)}\right|^2 = \frac{1}{2^n}\left(1 + 1 + \cdots - 1 - 1 - 1 \cdots\right)$$

$$\left(\frac{1}{2^n}(0)\right)^2$$

$$= \begin{cases} \left(\frac{1}{2^n} 2^n\right)^2 = 1 & f(k)=0 \;\forall k \\ \left|\frac{1}{2^n}(-2^n)\right|^2 = |-1|^2 = 1 & f(k)=1 \;\forall k \\ 0 & f(k) = \begin{cases} 0 \\ 1 \end{cases} \end{cases}$$