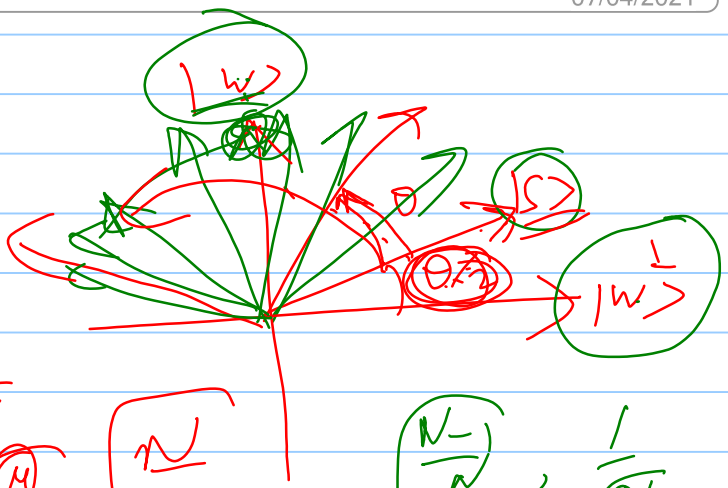
Lecture 6:

Note Title 07/04/2021

$r = 0$

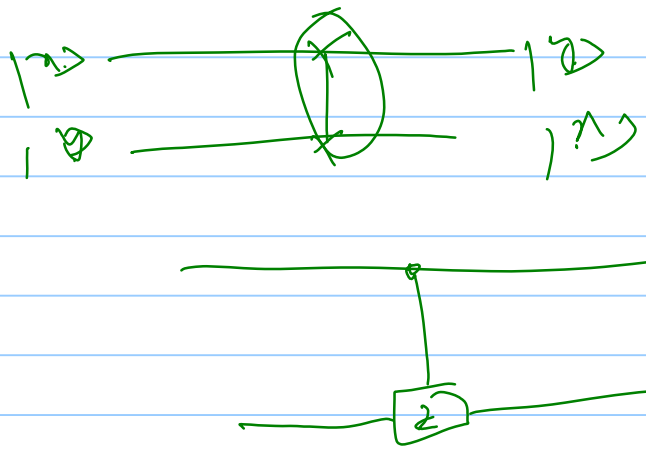
n is large



$\frac{1}{\sqrt{n}}$

$r \propto \sqrt{\frac{n}{M}}$

$\frac{N-1}{N}$, $\frac{1}{\sqrt{N}}$

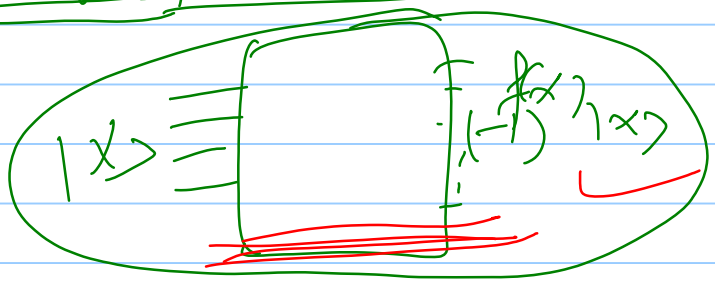


$\{|0\rangle, |1\rangle$
$|+\rangle, |-\rangle\}$

$\{\varphi', \varphi', \ldots\}$

$\{|+\rangle, |-\rangle$
$|+\rangle, |-\rangle\}$

- Bernstein – Vazirani
- Constructing phase oracles



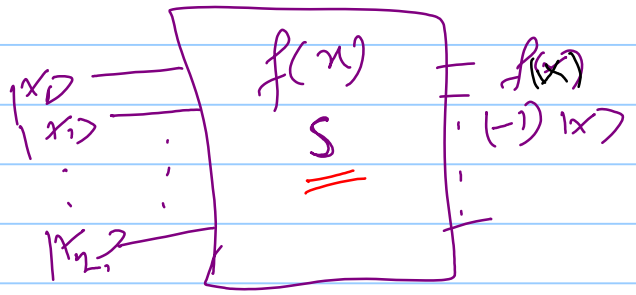
Bernstein – Vazirani

- Problem:

$S = s_0 s_1 \cdots s_{n-1}$

example: $S = 110$

x	$f(x)$
010	$0+0+0=0$
110	$1+1+0+0=0$

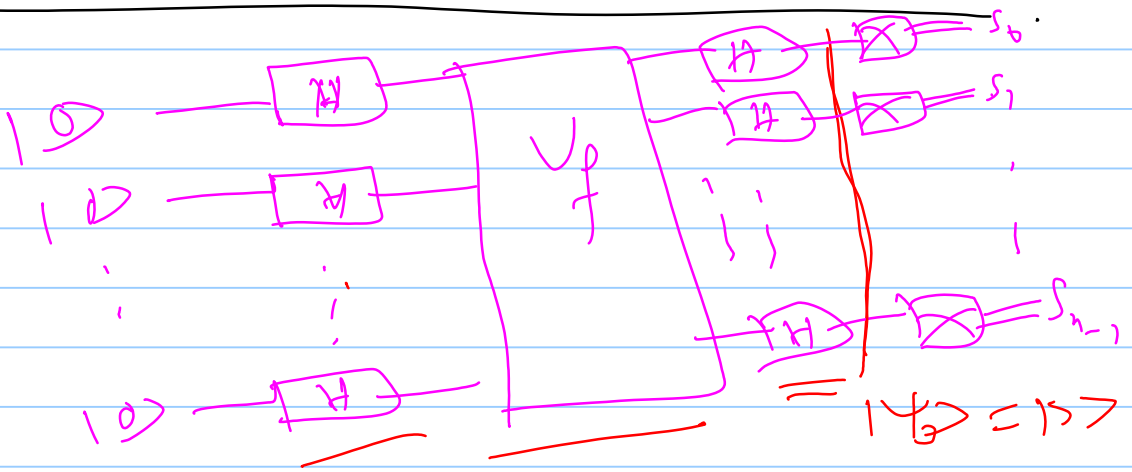


$f(x) = x \cdot S$

$|0010\rangle \longrightarrow |0010\rangle$

$|1110\rangle \longrightarrow |1100\rangle$

Goal: find S.



$N = 2^n$

(0) $|\psi_0\rangle = |0\rangle = |00\cdots0\rangle = |0\rangle^{\otimes n}$

(1) $|\psi_1\rangle = H^{\otimes n}|0\rangle$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{x\cdot 0} |x\rangle$$

$$\boxed{|\psi_1\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle}$$

(2) $|\psi_2\rangle = U_f |\psi_1\rangle$

$$= \frac{1}{\sqrt{N}} \sum_x U_f |x\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_x (-1)^{f(x)} |x\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{N}} \sum_x (-1)^{x\cdot s} |x\rangle$$

(3) $|\psi_3\rangle = H^{\otimes n} |\psi_2\rangle$

$$= \frac{1}{\sqrt{N}} \sum_x (-1)^{x\cdot s} H^{\otimes n} |x\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_x (-1)^{x\cdot s} \frac{1}{\sqrt{N}} \sum_k (-1)^{kx} |k\rangle$$

$$= \frac{1}{N} \sum_k \left(\sum_x (-1)^{x\cdot(s+k)} \right) |k\rangle$$

$$\sum_x (-1)^{x\cdot(S+K)} = \left((-1)^{0.0\cdots(S+K)} + (-1)^{0\cdots1\cdot(S+K)} + \cdots\right)$$

$S = 101$

$$S+K = 101+001 = 110$$

$$\Rightarrow \left((-1)^0 + (-1)^1 + (-1)^{\cdots}\right)$$

$$= 0$$

$$K = 001 \quad , \quad S = 110$$

$$S+K = 100$$

$$S = K \qquad \begin{matrix} S \\ 101 \end{matrix} \quad \begin{matrix} K \\ 101 \end{matrix} = 000$$

$$\sum_x (-1)^{x\cdot 001} = \sum_x (-1)^0$$

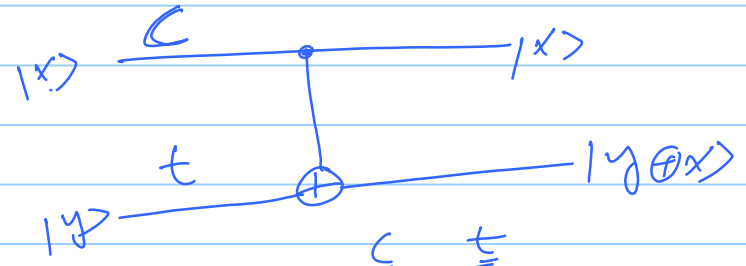
$$= \sum_{x=0}^{N-1} 1 = N$$

$$\boxed{|\psi_3\rangle = |S\rangle}$$

$$a|000\rangle + b|001\rangle + c|010\rangle + \cdots$$

$$|\psi_3\rangle = |101\rangle$$

How to Construct phase oracles:



$$C_x \overset{c}{|x\rangle} \overset{t}{|y\rangle} = |x\rangle |x \oplus y\rangle$$

$$C_x |0\rangle|0\rangle = |0\rangle|0\rangle$$

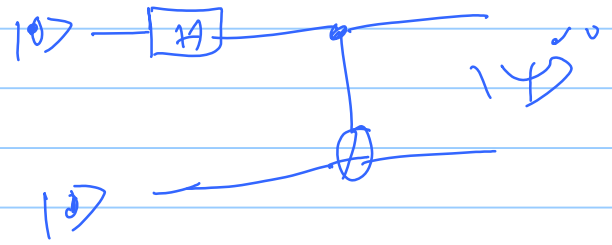
$$C_x |1\rangle|1\rangle = |1\rangle|0\rangle$$

$$C_x \overset{c}{|+\rangle}|0\rangle = C_x \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)(|0\rangle)$$

$$= C_x \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= |\Phi^+\rangle$$



$$C_x \overset{c}{|0\rangle} \overset{t}{|+\rangle} = C_x \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$= |0\rangle|+\rangle$$

$$C_x |0\rangle|-\rangle = |0\rangle|-\rangle$$

$$C_x |1\rangle|+\rangle = C_x \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$

$$= C_x \frac{1}{\sqrt{2}}(|11\rangle + |10\rangle)$$

$$= |1\rangle \otimes \frac{(|1\rangle + |0\rangle)}{\sqrt{2}}$$

$$= |1\rangle \otimes |+\rangle$$

$$C_x |1\rangle|-\rangle = \frac{1}{\sqrt{2}} C_x (|10\rangle - |11\rangle)$$

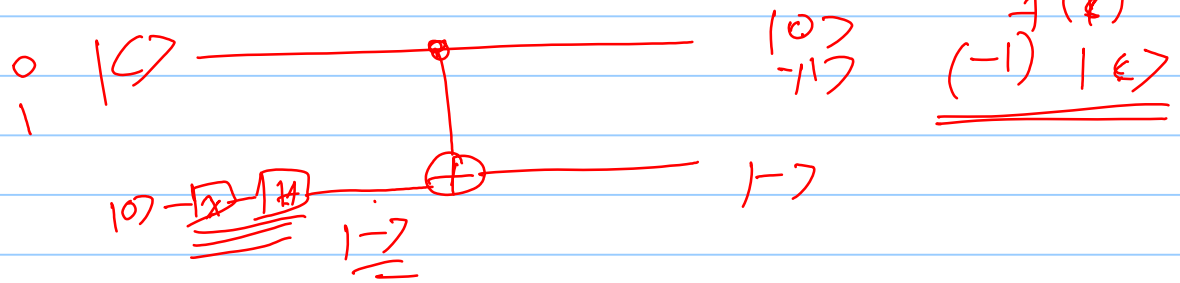
$$= \frac{1}{\sqrt{2}}(|11\rangle - |10\rangle)$$

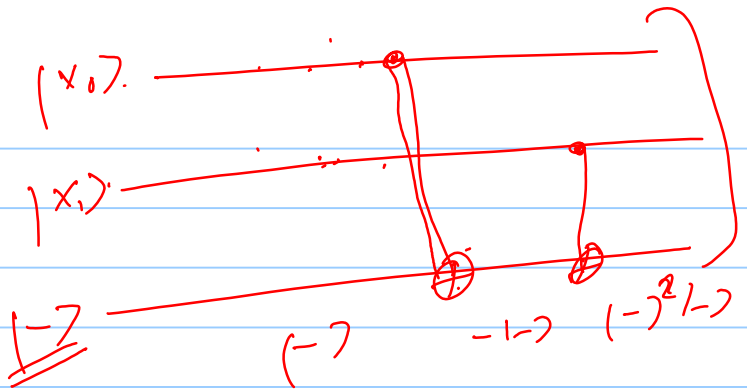
$$= |1\rangle \otimes \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

$$= -|1\rangle \otimes |-\rangle$$

$$= -|1\rangle|-\rangle$$

$$C_x |0\rangle|-\rangle = |0\rangle|-\rangle$$





$|00\rangle \quad |00\rangle$

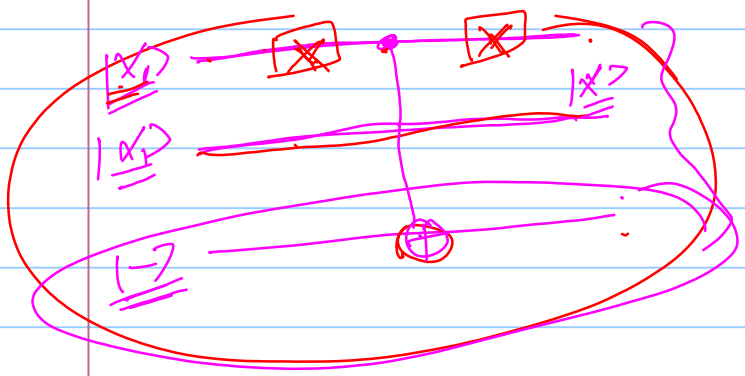
$|01\rangle \quad -|11\rangle$

$|10\rangle \quad -|10\rangle$

$|11\rangle \quad |11\rangle$

$$f(x) = \begin{cases} 0 & x = 00, \\ & x = 11 \\ 1 & x = 01, 10 \end{cases}$$

$|x\rangle \to (-1)^{f(x)}|x\rangle$



$f(00) \to 0$

$f(01) \to 0$

$f(10) \to 1$

$f(11) \to 1$

$|00\rangle \to |00\rangle$

$|01\rangle \to |01\rangle$

$|10\rangle \to -|10\rangle = (-1)^{f(x)}|x\rangle$

$|11\rangle \to -|11\rangle$

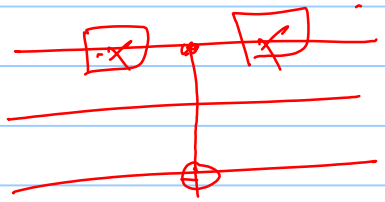
a $|11\rangle|10\rangle$

$(-)|x_0 x_1\rangle|-\rangle$

$|x_0 x_1\rangle|-\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)|-\rangle$

$$U|x_0 x_1\rangle \to = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

$S = 10$



$\|x\rangle$	$f(x)$	$(-1)^{f(x)}\|x\rangle$
$\|00\rangle$	0	$\|00\rangle$
$\|01\rangle$	0	$\|01\rangle$
$\|10\rangle$	1	$-\|10\rangle$
$\|11\rangle$	1	$-\|11\rangle$

~~-1 -1 -1 -1~~

a b c d e

0 1 2 3 4