Lecture 7. Fourier Transform

Note Title 08/04/2021

$$|q\rangle = \boxed{QFT|j\rangle \longrightarrow \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{\frac{2\pi i jk}{N}}|k\rangle}$$

n bits

$N = 2^n$; $|j\rangle = |j_1 j_2 \dots j_n\rangle$

$|j\rangle = |101\rangle$ $\qquad j = j_1 \times 2^{n-1} + j_2 \times 2^{n-2} + \dots j_n 2^0$

$j = 1\times 2^2 + 0\times 2 + 1\times 2^0 = 1\times 4 + 0 + 1 = 5$

$|x\rangle = |x_0 x_1 \dots x_{n-1}\rangle$ $\qquad |k\rangle = |k_1 k_2 \dots k_n\rangle$

$K = k_1 \times 2^{n-1} + k_2 \times 2^{n-2} \dots k_n \times 2^0$

$|q\rangle = QFT|j\rangle$ form orthonormal

$$\boxed{\langle q'|q\rangle = \delta_{qq'}}$$

$$|a\rangle = \sum a_q |q\rangle$$
$$= \sum$$

$$|\phi\rangle = \sum_{j=0}^{N-1} a_j |j\rangle$$

$$QFT|\phi\rangle = |\psi\rangle = \sum_{j=0}^{N-1} a_j \, QFT|j\rangle$$

$$= \sum_{j=0}^{N-1} a_j \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i jk}{N}} |k\rangle$$

$$\boxed{b_k = \frac{1}{\sqrt{N}} \sum a_j e^{\frac{2\pi i jk}{N}}} \quad DFT$$

$$= \sum_k \left(\frac{1}{\sqrt{N}} \sum a_j e^{\frac{2\pi i jk}{N}}\right)|k\rangle$$

a_j ... N

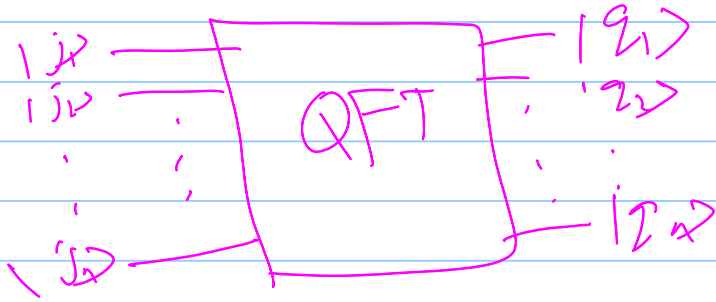
DFT a_j

$$= \sum_k b_k |k\rangle$$

Algorithm/Circuit for QFT

$|q\rangle = QFT|\underline{j}\rangle$

$|q_1 q_2 \cdots q_n\rangle = QFT|j_1 j_2 \cdots j_n\rangle$



$\boxed{N = 2^n}$

$$|q\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i \frac{jk}{N}} |k\rangle$$

$$|q\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left[2\pi i j \frac{1}{2^n} \sum_{l=1}^{N} k_l 2^{n-l}\right] |k\rangle$$

$$k = k_1 \times 2^{n-1} + k_2 \times 2^{n-2} \cdots k_n 2^0$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left[2\pi i j \sum_{l=1}^{N} k_l 2^{-l}\right] |k\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\bigotimes_{l=1}^{N} e^{2\pi i j k_l 2^{-l}} \right) |k\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \sum_{k_3=0}^{1} \cdots \sum_{k_n=0}^{1} \left(\bigotimes_{l=1}^{N} e^{2\pi i j k_l 2^{-l}} \right) |k_1 k_2 \cdots k_n\rangle$$

$|k_1\rangle \otimes |k_2\rangle \otimes \cdots$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \bigotimes_{l=1}^{N} \sum_{k_l=0}^{1} \left(e^{2\pi i j k_l 2^{-l}} |k_l\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \left(\sum_{k_1=0}^{1} \left(e^{2\pi i j k_1 2^{-1}}\right)|k_1\rangle\right) \otimes \left(\sum_{k_2=0}^{1} e^{2\pi i j k_2 2^{-2}} |k_2\rangle\right)$$

$$\cdots \otimes \sum_{k_n=0}^{1} e^{2\pi i j k_n 2^{-n}} |k_n\rangle$$

$$|q\rangle = \underbrace{\frac{|0\rangle + e^{2\pi i j 2^{-1}}|1\rangle}{\sqrt{2}}}_{|q_1\rangle} \otimes \underbrace{\frac{|0\rangle + e^{2\pi i j 2^{-2}}|k_2\rangle}{\sqrt{2}}}_{|q_2\rangle}$$

$$\otimes \frac{|0\rangle + e^{2\pi i j 2^{-3}}|k_3\rangle}{\sqrt{2}} \otimes \cdots \otimes \frac{|0\rangle + e^{2\pi i j 2^{-n}}|1\rangle}{\sqrt{2}}$$

$$|q\rangle = |q_1 q_2 \cdots q_n\rangle$$

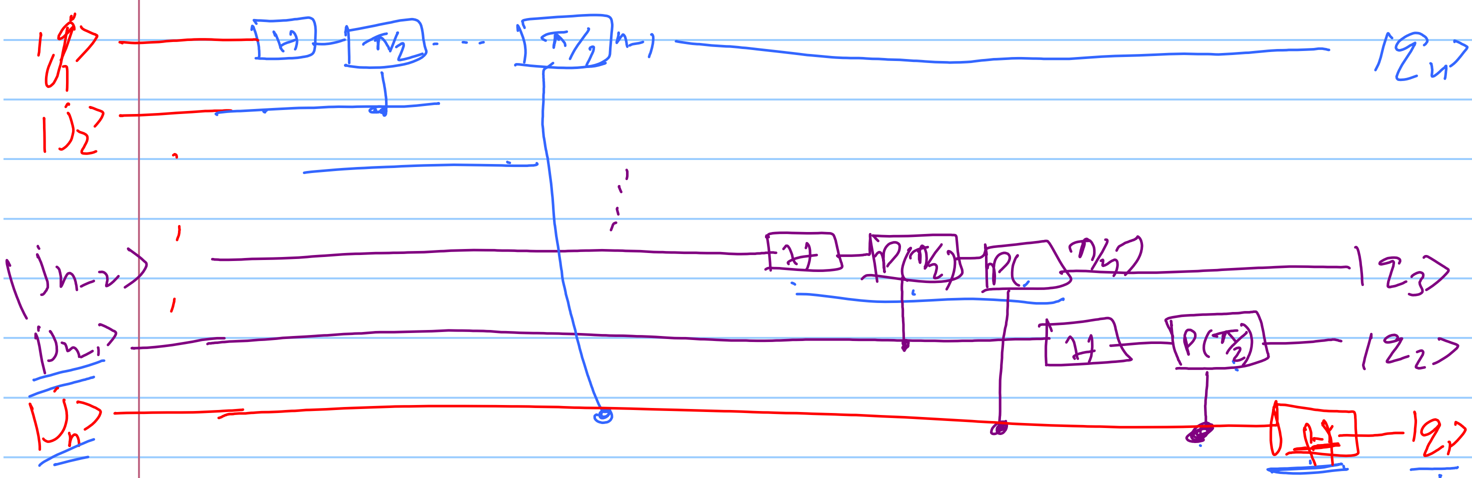
$$|q_1\rangle = \frac{|0\rangle + e^{2\pi i \frac{j}{2}}|1\rangle}{\sqrt{2}} \qquad \boxed{e^{2\pi i m} = 1}$$

$$e^{2\pi i \frac{j}{2}} = e^{2\pi i \left[\frac{j_1 \times 2^{n-1}}{2} + \frac{j_2 \times 2^{n-2}}{2} + \frac{j_3 \times 2^{n-3}}{2} + \cdots \frac{j_{n-1} \times 2}{2} + \frac{j_n 2^0}{2}\right]}$$

$$= e^{2\pi i \frac{j_n}{2}} = \begin{cases} 1 & \text{if } j_n = 0 \\ -1 & j_n = 1 \end{cases}$$

$$|q_1\rangle = \frac{|0\rangle + e^{2\pi i \frac{j_n}{2}}|1\rangle}{2}$$

$$|q_1\rangle = \begin{cases} \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} & j_n = 0 \\[2ex] \dfrac{|0\rangle - |1\rangle}{\sqrt{2}} & j_n = 1 \end{cases}$$



$$|q_2\rangle = \frac{|0\rangle + e^{2\pi i \frac{j}{2^2}}|1\rangle}{\sqrt{2}}$$

$$e^{2\pi i \frac{j}{2^2}} = e^{2\pi i\left[\frac{j_1 \times 2^{n-1}}{2^2} + \frac{j_2 \times 2^{n-2}}{2^2} + \cdots + \frac{j_{n-1} \times 2^1}{2^2} + \frac{j_n \times 1}{2^2}\right]}$$

$$e^{2\pi i \frac{j}{2^2}} = e^{2\pi i\left[\frac{j_{n-1}}{2} + \frac{j_n}{2^2}\right]}$$

$$= e^{i\pi j_{n-1}} \; e^{i\frac{2\pi}{2^2} j_n} \qquad \phi = \frac{2\pi}{2^2} = \frac{\pi}{2}$$

$$e^{i\frac{\pi}{2} j_n}$$

$$|q_3\rangle = \frac{|0\rangle + e^{2\pi i \frac{j}{2^3}}|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + e^{2\pi i\left[\frac{j_{n-2}}{2} + \frac{j_{n-1}}{4} + \frac{j_n}{8}\right]}}{}$$

$$\sqrt{2}$$

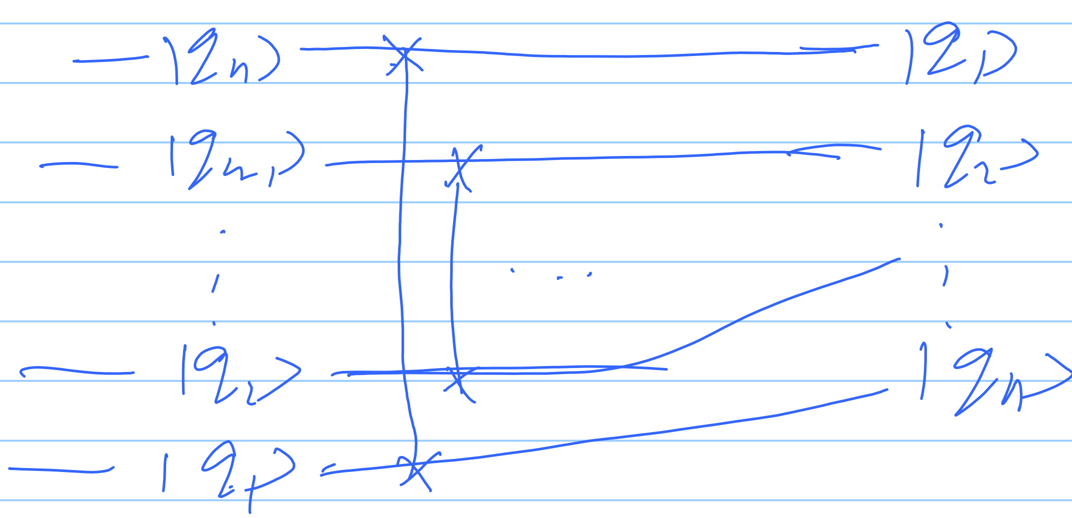
$$= |0\rangle + e^{i\pi j_{n-2}} \, e^{i\frac{\pi}{4} j_{n-1}} \, e^{i\frac{\pi}{4} j_n}$$

$$|q_n\rangle = \frac{|0\rangle + e^{2\pi i \frac{j}{2^n}} |1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + e^{2\pi i \left(\frac{j_1}{2} + \frac{j_2}{2^2} + \cdots \frac{j_n}{2^n}\right)} |1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle + e^{i\left(\pi j_1 + \frac{\pi}{2} j_2 + \cdots \frac{\pi}{2^{n-1}} j_n\right)} |1\rangle}{\sqrt{2}}$$

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$



Gate Count: $1+2+3+\dots n$

$$= \frac{n(n+1)}{2} + \frac{n}{2}$$

$$\boxed{O(\text{ops}) \approx n^2} \times 1\,\text{ms}$$

Classically:

$$O(n 2^n) \times 1\,\text{ms}$$

$n = 1$ million

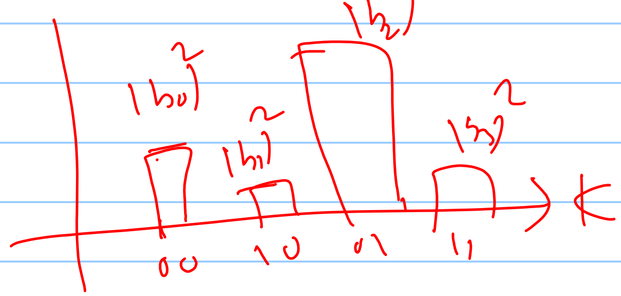
$$(10^6)^2 = 10^{12}$$

$$10^6 \times 2^{10^6}$$

$$\left(e^{iaI} X\right)^\dagger = X^\dagger e^{-ia}$$

$$|\phi\rangle = \sum a_j |k\rangle$$

$$|\psi\rangle = QFT|\phi\rangle = \sum b_k |k\rangle$$



$$= |b_k|^2$$