Note Title 09/04/2021

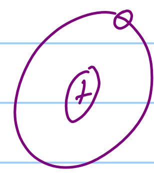
Lecture 8: QPE

Unitary operator U, QPE finds eigenvalues of U

$$U|\psi\rangle = a|\psi\rangle$$ if $|\psi\rangle$ is an eigenvector

$$U|u\rangle = a_u|u\rangle \longrightarrow \boxed{|a|^2 = 1}$$

$$U|u\rangle = e^{i\theta_u}|u\rangle \qquad \boxed{a_u = e^{i\theta_u}}$$

- , H — O — H $\Longrightarrow \boxed{\hat{H}|\psi\rangle = E_n|\psi\rangle}$

- To find number of solutions of search

$$r = \frac{N}{M} \quad \{a, b, c, \dots c\}$$

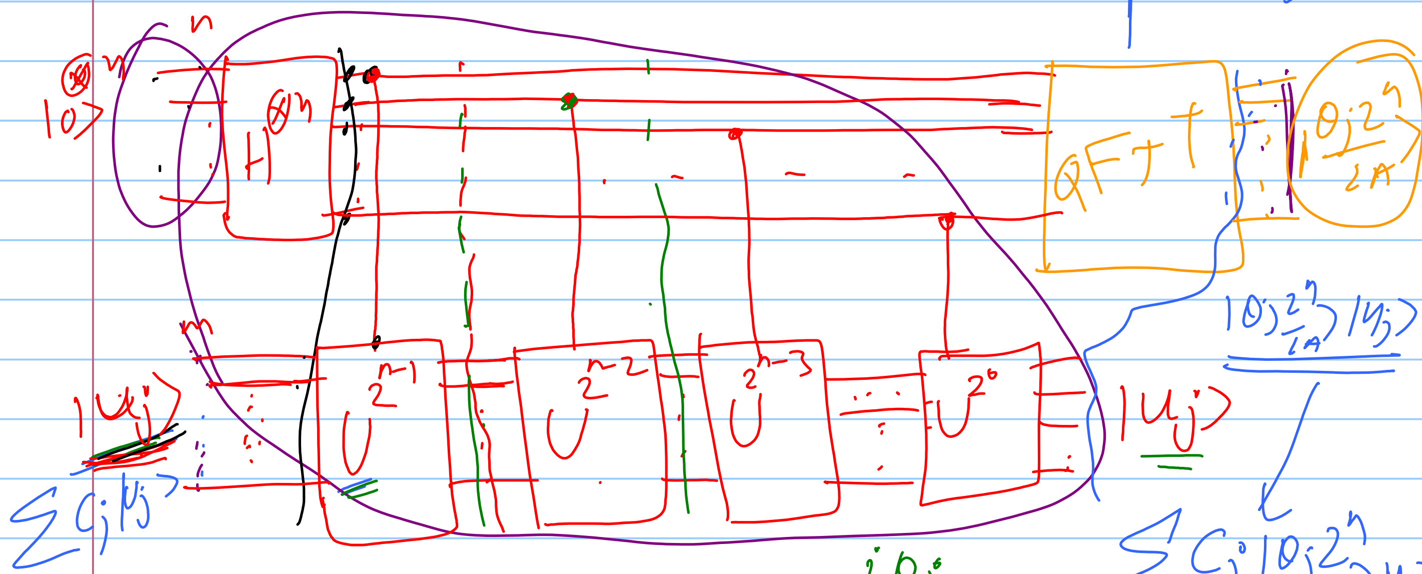
Q.C.

Quantum Phase Estimation

$$U|u_j\rangle = e^{i\theta_j}|u_j\rangle$$

$$|\psi\rangle = \sum_{j=1}^{M} c_j|u_j\rangle$$

$$U|\psi\rangle = \sum_{j=1}^{M} c_j e^{i\theta_j}|u_j\rangle$$



$$|\tfrac{\theta_j 2^n}{2\pi}\rangle|u_j\rangle$$

$$\sum c_j|\tfrac{\theta_j 2^n}{2\pi}\rangle|u_j\rangle$$

$$U|u_j\rangle = e^{i\theta_j}|u_j\rangle$$

$$U^x|u_j\rangle = U^{x-1}U|u_j\rangle$$
$$= e^{i\theta_j}U^{x-1}|u_j\rangle$$
$$= e^{i2\theta_j}U^{x-2}|u_j\rangle$$

$$\boxed{U^3|u_j\rangle = e^{i3\theta_j}|u_j\rangle}$$

$$\boxed{U^x|u_j\rangle = e^{ix\theta_j}|u_j\rangle}$$

$$\Rightarrow |0\rangle^{\otimes n}|u_j\rangle = |00\cdots0\rangle|u_j\rangle = |0\rangle|u_j\rangle$$

$$\Rightarrow H^{\otimes n}|0\rangle^{\otimes n}|u_j\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \dots \otimes (|0\rangle + |1\rangle) \otimes |u_j\rangle$$

$$(|0\rangle + |1\rangle) \otimes |u_j\rangle$$

$$\Rightarrow |0\rangle |u_j\rangle + |1\rangle |u_j\rangle$$

$$\Rightarrow |0\rangle |u_j\rangle + |1\rangle U^{2^{n-1}} |u_j\rangle$$

$$= |0\rangle |u_j\rangle + |1\rangle e^{i\theta_j 2^{n-1}} |u_j\rangle$$

$$= (|0\rangle + e^{i\theta_j 2^{n-1}} |1\rangle) \otimes |u_j\rangle$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^n (|0\rangle + e^{i\theta_j 2^{n-1}} |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \dots \otimes (|0\rangle + |1\rangle) \otimes |u_j\rangle$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^n (|0\rangle + e^{i\theta_j 2^{n-1}} |1\rangle) \otimes (|0\rangle + e^{i\theta_j 2^{n-2}} |1\rangle) \otimes (|0\rangle + e^{i\theta_j 2^{n-3}} |1\rangle)$$

$$\otimes \dots \otimes (|0\rangle + e^{i\theta_j 2} |1\rangle) \otimes (|0\rangle + e^{i\theta_j} |1\rangle) \otimes |u_j\rangle$$

$$\left(\frac{1}{\sqrt{2}}\right)^n \left(|0\rangle + e^{\frac{2\pi i j}{2}} |1\rangle\right) \otimes \left(|0\rangle + e^{\frac{2\pi i j}{2^2}} |1\rangle\right) \otimes \dots \otimes \left(|0\rangle + e^{\frac{2\pi i j}{2^n}} |1\rangle\right)$$

$$\boxed{2\pi j = \theta_j 2^n}$$

$$|j\rangle$$

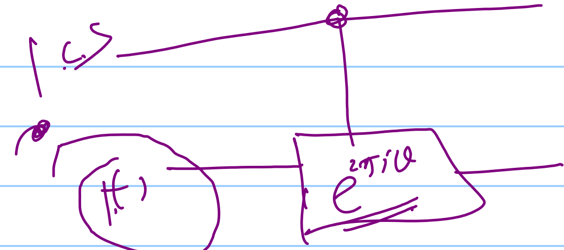
$$7 \leftarrow |111\rangle$$

$$2^3 = 8$$

$$j = \frac{\theta_j 2^n}{2\pi}$$

Output: $\left|\frac{\theta_j 2^n}{2\pi}\right\rangle \Rightarrow \boxed{\theta_j = \frac{\text{output}}{2^n} \times 2\pi}$

$C(U)$



$$\underline{U} = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i\theta} \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$1 \to e^{i\theta}$$

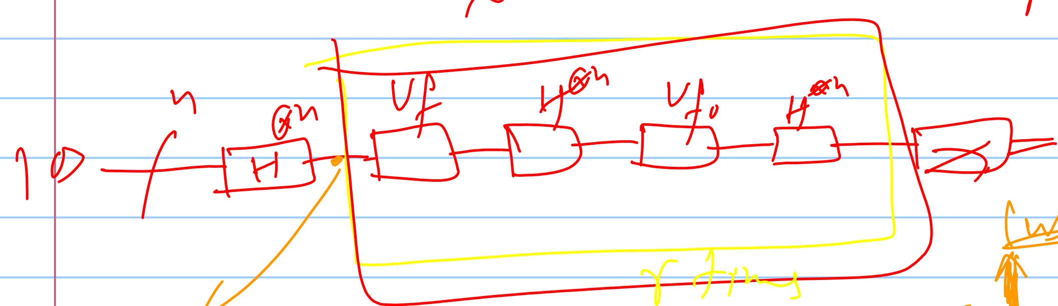
$$\boxed{\underline{U}^m = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i m\theta} \end{bmatrix}}$$

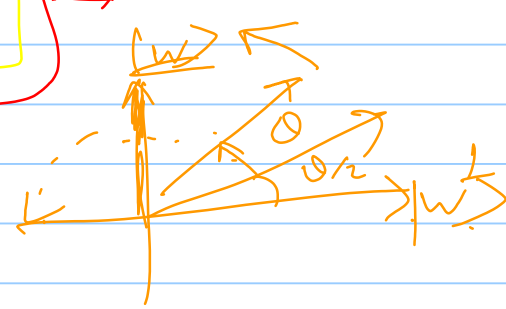
Quantum Counting!

$$\underline{N} = 2^n$$

$$\underline{r} \propto \sqrt{\frac{N}{M}} \qquad \{a, b, \dots\}$$

M number of solutions





$$\underline{|s\rangle} = \cos\frac{\theta}{2}|w^\perp\rangle + \sin\frac{\theta}{2}|w\rangle$$

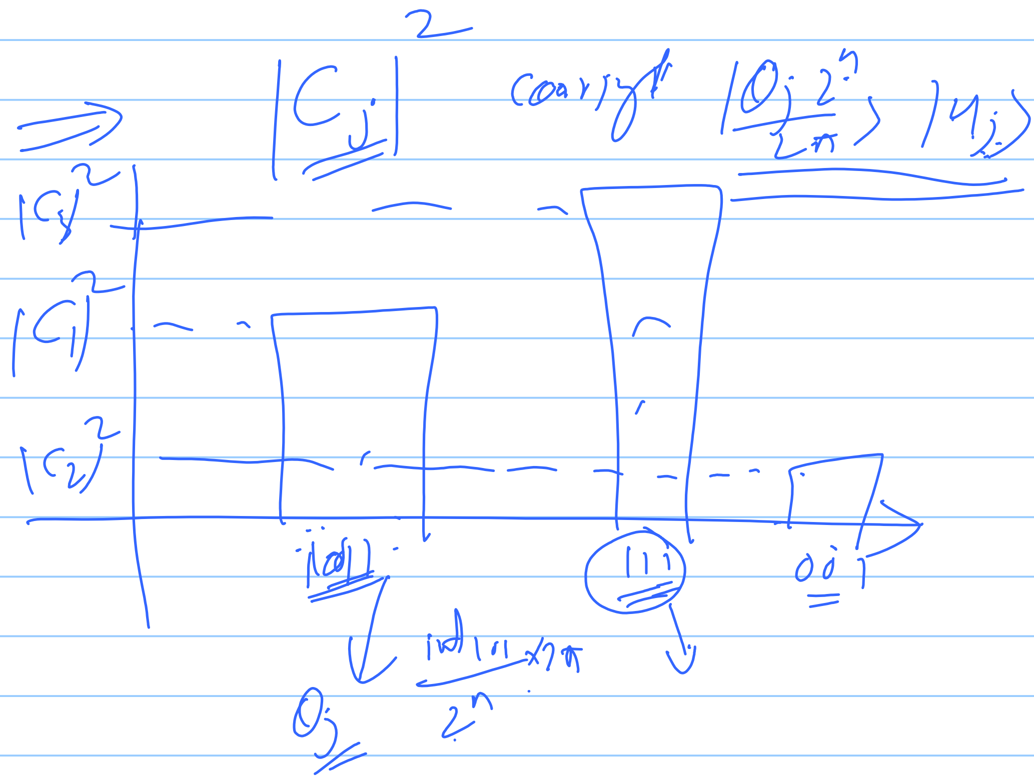
$$\underline{G|s\rangle} = \underline{\cos\left(\theta + \frac{\theta}{2}\right)}|w^\perp\rangle + \underline{\sin\left(\theta + \frac{\theta}{2}\right)}|w\rangle$$

$$\underline{G} = \underline{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix}}$$

$$\boxed{\sin\frac{\theta}{2} = \sqrt{\frac{N}{M}}}$$

$$e_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$e^{i\theta} \qquad e^{-i\theta} = e^{i(2\pi - \theta)}$$



Q.C.